Examples of Group Representations

Trivial Representation: $\varphi : G \longrightarrow GL_1(\mathbb{C}) = \mathbb{C}^\times$

by $\varphi_g = 1$ for all $g \in G$

Warning: $V = \{0\} = \mathbb{C}^0 = 0$, $\dim V = 0$

$$GL(V) = GL_0(\mathbb{C}) = \{id\}$$

$\varphi : G \longrightarrow GL(0)$, "0 representation"

Example: $G = \mathbb{Z}_4$

$$\varphi : \mathbb{Z}_4 \longrightarrow GL_1(\mathbb{C}) = \mathbb{C}^\times$$

$$\varphi_{[k]} := e^{2\pi i k/4} = i^k, \qquad \varphi_{[0]} = 1 \quad \varphi_{[2]} = -1 \quad \varphi_{[1]} = i \quad \varphi_{[3]} = -i$$

$$\psi : \mathbb{Z}_4 \longrightarrow GL_2(\mathbb{C})$$

$$\psi_{[k]} := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^k$$

Ex: $G = \mathbb{Z}_n, \quad m \in \mathbb{Z}$

$$\varphi : \mathbb{Z}_n \longrightarrow GL_1(\mathbb{C}) = \mathbb{C}^\times$$

$$\varphi_{[k]} := e^{2\pi i k m / n} = \zeta^{km}, \quad \zeta = e^{2\pi i / n}$$

$$\psi : \mathbb{Z}_n \longrightarrow GL_d(\mathbb{C}), \qquad m_1, \dots, m_d \in \mathbb{Z}$$

$$\psi_{[k]} := \begin{bmatrix} \zeta^{km_1} & & 0 \\ & \ddots & \\ 0 & & \zeta^{km_d} \end{bmatrix}$$

Standard representation of S_n:

$$\rho : S_n \longrightarrow GL_n(\mathbb{C})$$

$$\rho_g := \begin{bmatrix} e_{g(1)} & \cdots & e_{g(n)} \end{bmatrix} \quad \text{– permutation matrix}$$

e.g. $\rho : S_3 \longrightarrow GL_3(\mathbb{C})$

$$\rho_{(1\,2)} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \rho_{(1\,2\,3)} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Given

$$\begin{tikzcd} H \arrow[r, "\psi", "\text{homomorphism}"'] \arrow[rr, bend right, "\varphi\circ\psi"'] & G \arrow[r, "\varphi"] & GL(V) \end{tikzcd}$$

$\varphi\circ\psi$ – representation of H.

Ex:

$$\begin{tikzcd} S_n \arrow[r, "\mathrm{sgn}"] \arrow[rr, bend right, "\mathrm{sgn}"'] & \{\pm 1\} \arrow[r, hook, "\varphi"] & GL_1(\mathbb{C}) = \mathbb{C}^\times \end{tikzcd}$$

"sign representation"

If $H \leq G$

$$\begin{tikzcd} H \arrow[r, hook] \arrow[rr, bend right] & G \arrow[r, "\varphi"] & GL(V) \end{tikzcd}$$

$\varphi|_H :=$ restriction of φ to H.

Ex: $V = \{ \text{functions } f : \mathbb{R} \to \mathbb{R} \} \ni f$

$f+g, \quad \lambda f \qquad$ vector space over $\mathbb{R}$

$$\varphi : \mathbb{Z}_2 \longrightarrow GL(V)$$

$$\varphi_{[k]}(f) := \left[\underset{\substack{\cap \\ \mathbb{R}}}{x} \longmapsto f((-1)^k x) \right]$$

$\{ f \in V \mid \varphi_{[1]}(f) = f \} =$ "even functions"

$\{ f \in V \mid \varphi_{[1]}(f) = -f \} =$ "odd functions"